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TITLE OPTICAL ANALOGS OF MODEL ATOMS IN FIELDS

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Optical Analogs of Model Atoms in Fields

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"Quantum mechanics is not a bad preparation for optics." - Dennis Gabor

Abstract

The equivalence of the paraxial wave equation to a time-dependent Schrödinger equation is exploited to construct optical analogs of model atoms in monochromatic fields. The approximation of geometrical optics provides the analog of the corresponding classical mechanics. Optical analogs of Rabi oscillations, photoionization, stabilization, and the Kramers-Henneberger transformation are discussed. One possibility for experimental realization of such optical analogs is proposed. These analogs may be useful for studies of "quantum chaos" when the ray trajectories are chaotic.

Everyone knows there are many analogies between optics and quantum mechanics. I would like to suggest here that some recent effects of interest for atoms in strong fields might possibly be realized in the propagation of light.

First recall that the time-dependent Schrödinger equation is formally the same as the paraxial wave equation of optics. This is well known to optical physicists. (See, for instance, Cook [1] or Stoler [2].) We will briefly derive this correspondence, with the slight generalization of allowing the refractive index to vary both axially and transversely. This will lead us to an optical analog of an atom in a monochromatic field.

Assume a linearly polarized monochromatic field with electric field amplitude $E(\mathbf{r})e^{-i\omega t}$. E satisfies the Helmholtz equation, $\nabla^2 E + k^2 n^2 E = 0$, where $k \equiv \omega/c$ and n is the refractive index. Write $E(\mathbf{r}) = E_o(\mathbf{r}_\perp, z)e^{ikz}$, where E_o is assumed to be slowly varying in z compared with e^{ikz} ; \mathbf{r}_\perp is the coordinate in the xy plane, perpendicular to the direction of propagation z . Then we can drop $\partial^2 E_o / \partial z^2$ compared with $k \partial E_o / \partial z$ in the Helmholtz equation and work with the paraxial wave equation

$$2ik \frac{\partial E_o}{\partial z} = -\nabla_\perp^2 E_o - k^2(n^2 - 1)E_o, \quad (1)$$

where $\nabla_{\perp}^2 \equiv (\partial^2/\partial x^2 + \partial^2/\partial y^2)$. Obviously the paraxial wave equation has the same form as the Schrödinger equation for a particle constrained to move in two spatial dimensions:

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla_{\perp}^2 \psi + V\psi. \quad (2)$$

It is convenient to scale the time and space variables. Introduce in (2) the dimensionless time $\tau = (\hbar/ma^2)t \equiv \omega_o t$ and the dimensionless coordinate variables $X = x/a, Y = y/a$, where a is some convenient length. Then (2) becomes

$$i \frac{\partial \psi}{\partial \tau} = -\frac{1}{2} \bar{\nabla}_{\perp}^2 \psi + \frac{V}{\hbar\omega_o} \psi, \quad (3)$$

where $\bar{\nabla}_{\perp}^2 = a^2 \nabla_{\perp}^2$ is the transverse Laplacian in the X, Y variables. Similarly introduce the dimensionless coordinate variables $Z = z/kb^2, X = x/b, Y = y/b$, where b is some convenient length scale for the optics problem, so that (1) becomes

$$i \frac{\partial E_o}{\partial Z} = -\frac{1}{2} \bar{\nabla}_{\perp}^2 E_o - \frac{1}{2} k^2 b^2 (n^2 - 1) E_o. \quad (4)$$

The effective potential in this “optical Schrödinger equation” is thus

$$V_{\text{opt}} \equiv -(\hbar\omega_o) \frac{1}{2} k^2 b^2 (n^2 - 1). \quad (5)$$

To keep things really simple we will consider a few one-dimensional examples, where equations (3) and (4) become respectively

$$i \frac{\partial \psi}{\partial \tau} = -\frac{1}{2} \frac{\partial^2 \psi}{\partial X^2} + \frac{1}{\hbar\omega_o} V(Xa, \frac{\tau}{a}) \psi \quad (6)$$

$$i \frac{\partial E_o}{\partial Z} = -\frac{1}{2} \frac{\partial^2 E_o}{\partial X^2} + \frac{1}{\hbar\omega_o} V_{\text{opt}}(Xb, \frac{1}{2} kb^2 Z) E_o. \quad (7)$$

Classical (Ray) Limit

The classical Newton equation of motion corresponding to (6) is $m\ddot{x} = -\partial V/\partial x$. If we use (7) to construct the corresponding “optical Newton equation,” we obtain $d^2x/dz^2 \cong \partial n/\partial x$ for $n \cong 1$. This is the paraxial approximation to the ray equation $(d/ds)(n d\mathbf{r}/ds) = \nabla n$ for a position vector \mathbf{r} of a point on a ray, with s a distance measured along the ray. [3]

This example brings out a simple but important point. In the approximation of geometrical optics one deals with *families* of rays, for a ray merely gives us some information about a point on the wavefront. In the same way the comparison of classical and quantum theories must involve an *ensemble* of trajectories. This is well known, of

course, but sometimes it seems to be forgotten by those who express great surprise at the fact that classical systems can exhibit chaos (in the sense of a positive Lyapunov exponent), while the corresponding quantum systems do not. This can be understood in part from the simple consequence of Liouville's theorem that classical *distributions* of trajectories cannot exhibit the "very sensitive dependence on initial conditions" that is the hallmark of classical chaos. [4]

The Harmonic Oscillator

According to equations (5)-(7) we can produce the optical analog of a harmonic oscillator by making $V_{\text{opt}}/\hbar\omega_0 = -\frac{1}{2}k^2b^2(n^2 - 1) = CX^2b^2$, or $n(x) = 1 - Dx^2$ for $n \cong 1$, where C, D are positive constants. This defines a lenslike medium, [5] also known as a *graded-index waveguide*. In this case the "optical Newton equation" above becomes $d^2x/dz^2 + Dx = 0$, so that the ray displacements oscillate harmonically with propagation in the lenslike medium. In the wave-optical description, as in the quantum-mechanical oscillator, the modes of the field in the lenslike medium are such that $E_0(x)$ is a Hermite polynomial, the lowest-order mode being the ubiquitous Gaussian beam. Since spherical mirrors affect rays in basically the same way as a thin lens, it should come as no surprise to anyone who knows elementary quantum mechanics that the modes of stable laser resonators are Gaussian fields.

Anharmonic Oscillator

Similarly we can construct the optical analog of an anharmonic oscillator with potential $V(x) \propto x^2 + Ax^4$, say, by choosing $n(x) = 1 - D(x^2 + Ax^4)$. By choosing the transverse index variation $n(x)$ appropriately we can "design" any anharmonic oscillator we like.

Time-Dependent Potential

Thus far we have taken n to be a function of x but not z . That is, we have not allowed the refractive index to vary along the direction of propagation. For the quantum problem, this means we have restricted ourselves to time-independent potentials. To develop optical analogs of driven quantum systems with time-dependent perturbations, we now allow n to be a function of both x and z .

A potential that has been used recently in numerical experiments on above-threshold ionization and high-order harmonic generation is [6]

$$V(\tau, t) = \frac{-e^2}{\sqrt{x^2 + a^2}} - exA \cos \omega t, \quad (8)$$

or

$$\frac{1}{\hbar\omega_0} V(Xa, \frac{\tau}{a}) = -\frac{1}{\sqrt{X^2 + 1}} - XF \cos \mu\tau, \quad (9)$$

where we choose $\hbar\omega_o = e^2/a_o$, $a = a_o$, $\mu = \omega/\omega_o$, and $F = A(e/a_o^2)^{-1}$ is the field strength in atomic units. The optical analog of the Schrödinger equation with this potential is obtained when

$$n^2(x, z) - 1 = \frac{2}{k^2 b^2} \left[\frac{1}{\sqrt{x^2/b^2 + 1}} + \frac{x}{b} F \cos\left(\frac{2\mu z}{kb^2}\right) \right], \quad (10)$$

in which case (7) becomes

$$i \frac{\partial E_o}{\partial Z} = -\frac{1}{2} \frac{\partial^2 E_o}{\partial X^2} + U(X) E_o - X F \cos(\mu Z) E_o. \quad (11)$$

with $U(X) \equiv -(X^2 + 1)^{-1/2}$. Numerical solutions of such a one-dimensional Schrödinger equation go back at least as far as Goldberg, Schey, and Schwartz. [7] More recently such numerical solutions have been used in studies of quantum chaos [8] and above-threshold ionization. [6,9]

Since paraxial wave propagation with a periodically varying index like (10) is described by a Schrödinger equation for a particle in a time-independent potential plus a sinusoidal applied field, we can do all the usual things done to treat atoms in fields. For instance, we can use an expansion in basis states: Consider solutions of the "unperturbed" system defined by

$$i \frac{\partial E_o}{\partial Z} = -\frac{1}{2} \frac{\partial^2 E_o}{\partial X^2} + U(X) E_o. \quad (12)$$

(We will assume $U(-X) = U(X)$.) Write $E_o(X, Z) = g(X) e^{-iKZ}$, so that $g(X)$ satisfies the eigenvalue equation

$$-\frac{1}{2} \frac{d^2 g}{dX^2} + U(X) g(X) = K g(X). \quad (13)$$

The solutions $g_n(X)$ with eigenvalues K_n define the optical modes for the unperturbed paraxial wave equation (12). To solve (11) we can write

$$E_o(X, Z) = \sum_n a_n(Z) g_n(X) e^{-iK_n Z}, \quad (14)$$

and then (11) implies

$$i \frac{da_n}{dZ} = -F \cos(\mu Z) \sum_m X_{nm} e^{-i(K_m - K_n)Z} a_m(Z), \quad (15)$$

$$X_{nm} \equiv \int_{-\infty}^{\infty} dX g_n^*(X) X g_m(X). \quad (16)$$

Thus the mode amplitudes $a_n(Z)$ satisfy the same equations as the probability amplitudes of the corresponding quantum-mechanical problem.

Two-State Atom

Suppose the field is propagating in the lowest-order g_1 mode and encounters a sinusoidal perturbation along Z , such that $\mu \cong K_2 - K_1$ in equation (15). If $|K_n - K_1|$ and $|K_n - K_2|$ for $n > 2$ are sufficiently different from μ that all the $a_n(Z)$ obtained from (15) with $n > 2$ are very small, we can approximate (15) by

$$\begin{aligned} i \frac{da_1}{dZ} &= -F \cos(\mu Z) X_{12} e^{-i(K_2 - K_1)Z} a_2(Z) \\ &\cong -\frac{1}{2} X_{12} F a_2(Z), \end{aligned} \quad (17)$$

$$i \frac{da_2}{dZ} \cong -\frac{1}{2} X_{21} F a_1(Z), \quad (18)$$

if we assume $\mu = K_2 - K_1$. (The approximation made in writing these equations will be recognized as the "rotating-wave approximation.") Thus

$$\frac{d^2 a_j}{dZ^2} + \frac{1}{4} |X_{12}|^2 F^2 a_j(Z) = 0, \quad j = 1, 2. \quad (19)$$

This means that, if we have a sinusoidal index variation along the direction of propagation, such that (11) applies, and if the spatial frequency of this variation equals the difference $K_2 - K_1$ between the eigenvalues of the modes $g_2(X)e^{-iK_2 Z}$ and $g_1(X)e^{-iK_1 Z}$, then as the field propagates it will oscillate between these two modes at the "Rabi frequency" $\frac{1}{2}|X_{12}|F$. Obviously we can construct "optical Bloch equations" from (17) and (18).

One-Dimensional Hydrogen Atom

Now let us return to the optical analog (11) of a "one-dimensional hydrogen atom." Suppose that a wave corresponding to the lowest-order mode $g_1(X)$ is launched into a medium with refractive index (10). If $F = 0$ such a wave will, ideally, propagate as $g_1(X)e^{-iK_1 Z}$. (For the present discussion we can consider any superposition of eigenmodes, but for simplicity let us just assume the lowest-order mode corresponding to the "ground state" of the atom in the quantum analog. We will also ignore "turn-on" effects associated with the sinusoidal perturbation.) If $F \neq 0$ the sinusoidal perturbation can cause "transitions" among modes, as we have just seen for the optical analog of a two-state atom. But in addition to transitions caused by such resonances between the spatial frequency of the index variation and the difference between two mode wave numbers, we can induce transitions at any spatial frequency of the index variations if the amplitude of these variations is large enough. For sufficiently large F we can have "photoionization": the optical analog is a transverse spreading of the wave into the "continuum," i.e., a transition in which at least part of the field is no longer associated with a transversely confined propagating mode.

There are obviously optical analogs of above-threshold ionization, harmonic generation, and stabilization. Let us consider the latter phenomenon. To do this we will first derive the optical analog of the Kramers-Henneberger (KH) transformation.¹

We can cast (11) in the language of bras, kets, and linear vector spaces of square-integrable functions:

$$i \frac{\partial}{\partial Z} |E_0\rangle = \left[\frac{1}{2} P^2 + U(X) - X F \cos(\mu Z) \right] |E_0\rangle, \quad (20)$$

with $[X, P] = i$. Write $F \cos(\mu Z) = -\partial A / \partial Z$ and define²

$$|E'_0\rangle = e^{-iP \int_0^Z dZ' A(Z') + (i/2) \int_0^Z dZ' A^2(Z')} e^{iX A(Z)} |E_0\rangle. \quad (21)$$

Then it follows from (20) that

$$i \frac{\partial}{\partial Z} |E'_0\rangle = \left[\frac{1}{2} P^2 + U(X + X_R(Z)) \right] |E'_0\rangle, \quad (22)$$

or

$$i \frac{\partial E'_0}{\partial Z} = -\frac{1}{2} \frac{\partial^2 E'_0}{\partial X^2} + U(X + X_R(Z)) E'_0, \quad (23)$$

where $X_R(Z)$, satisfying $d^2 X_R / dZ^2 = F \cos(\mu Z)$, is the ray displacement determined by geometrical optics with only the z -dependent part of the refractive index. The transformation from (20) to (22) is the "optical K-H transformation." Equation (23) is equivalent to

$$2ik \frac{\partial E'_0}{\partial z} = -\frac{\partial^2 E'_0}{\partial x^2} - k^2 [n_0^2(x + x_R(z)) - 1] E'_0, \quad (24)$$

where n_0 is the z -independent part of the refractive index and $x_R(z)$ is the geometrical ray displacement *without* this part of the index.

Stabilization

By analogy to the atomic case, stabilization can occur if F is large and if a high-(spatial) frequency approximation is justified. In the high-frequency approximation we replace $n_0^2(x + x_R(z)) - 1$ by its average over z . Then in effect the z -dependent part of the index is removed, and there is nothing left to cause transitions among different modes. In particular, the field will remain confined transversely to a large degree.

¹As I learned in a lecture by M.H. Mittleman (Los Alamos National Laboratory, April 3, 1991), this transformation may be traced at least as far back as W. Pauli and M. Fierz, *Nuovo Cim.* 15, 167 (1938). In fact a very similar transformation appears in a primarily relativistic setting in F. Bloch and A. Nordsieck, *Phys. Rev.* 52, 54 (1937).

²The reader may recognize the transformation from $|E_0\rangle$ to $|E'_0\rangle$ as a Power-Zienau transformation followed by the KH transformation.

Can These Optical Analogs be Constructed?

It is not difficult to imagine fabricating a medium with any transverse index variation we like, so let us assume this can be done. (It can pretty much be done with quantum wells, for instance.) The difficult part of producing an index variation like (10) is to get the part varying as $x \cos(Az)$.

Here is one way this might be done. Recall the paraxial ray equation $d^2x/dz^2 = \partial n / \partial x$ and suppose we bend the propagation path in such a way that the optical axis ($x = 0$) is displaced by $y(z)$. Then x is transformed to $x + y(z)$ and the paraxial ray equation is transformed to $d^2(x + y)/dz^2 = \partial n / \partial x$, or

$$\frac{d^2x}{dz^2} - \frac{\partial n}{\partial x} = -\frac{d^2y}{dz^2} \quad (25)$$

if $dx/dz, dy/dz$ are not too large. This is the equation one gets with index $n(x) - x d^2y/dz^2$. In other words, the index is changed from $n(x)$ to $n(x) + C(z)x$, where $C(z)$ is the curvature of the guiding optical axis. In particular,

$$n(x, z) = n(x) + A^2 x \cos(Az) \quad (26)$$

if $y(z) = \cos(Az)$. This is exactly the type of index we need to realize the optical analog of a model atom in a monochromatic field.

Note that a simple sinusoidal variation with z of the index, as in a distributed feedback laser,³ for instance, is not enough; we require a sinusoidal variation times x . This occurs when we bend the guiding axis as just described. Of course we should not bend the axis so strongly that the paraxial approximation breaks down, or that the boundaries of the medium intercept the beam.

Remarks

The simplified discussion just given is sufficient to bring out a few important points. One of these is that phenomena such as stabilization are not distinctly quantum-mechanical, for we have seen that they can be realized with *classical* waves. It should also be clear that the same sorts of conclusions inferred from the one-dimensional model carry over to the real case of two-dimensional transverse beam variations.

As mentioned earlier, the optical analogy might be useful in studies of "quantum chaos." If the index variations are such that the ray trajectories are chaotic, then the corresponding classical-mechanical system will be chaotic. The question of how this chaos might manifest itself quantum mechanically is then mathematically identical to the question of how the chaos of geometrical rays might manifest itself in wave propagation.

³Of course the distributed feedback case involves both backward- and forward-propagating waves, under conditions for which the paraxial approximation is not applicable.

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